ON WEAKLY MIXING MARKOV PROCESSES

BY

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ABSTRACT

A result of England and Martin on weak mixing (see [6]) is extended to Markov Processes in a strengthened form, and also to continuous time Markov Processes.

1. Definitions and notation

We refer to [3] for notation and basic results. Let (X, Σ, λ, P) be a Markov Process, that is, (X, Σ, λ) is a σ-finite measure space and P is a positive linear contraction on L_1 (i.e., P carries non-negative functions to non-negative functions). L_1 can be identified (via the Radon–Nykodim theorem) with the Banach space of finite signed measures absolutely continuous with respect to λ, and P acts on that space in the following manner: $\mu P(A) = \int P\mathbf{1}_A d\mu$ for $\mu \ll \lambda$, $A \in \Sigma$.

Throughout this paper λ is an invariant probability measure: $\lambda(X) = 1$ and $\lambda P = \lambda$. Note that all inequalities between functions and all set inclusions are to be understood in a λ-a.e. sense. Put $\Sigma_i(P) = \{A \in \Sigma; P\mathbf{1}_A = \mathbf{1}_A\}$; the process is called ergodic if $\Sigma_i(P) = \{\emptyset, X\}$.

Since λ is an invariant measure, P on L_∞ can be extended to contraction on L_1, and hence defines a contraction on L_2, also denoted by P, ([3], Chapter VII). P can be extended in a natural manner to the space of measurable square-summable complex valued functions on X (to be denoted henceforth by H), by putting $P(f + ig) = Pf + iPg$, $f, g \in L_2$.

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2. Weakly mixing processes

DEFINITION. P is *weakly mixing* if, for every finite measure $\mu \ll \lambda$ and every $A \in \Sigma$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \mu P^n(A) - \mu(X)\lambda(A) \right| = 0.$$

It is easy to verify that P is weakly mixing if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \langle P^n h_1, h_2 \rangle - \langle h_1, \mathbf{1} \rangle \langle \mathbf{1}, h_2 \rangle \right| = 0$$

for every $h_1, h_2 \in H$.

Let us quote two results, to be used in the sequel.

RESULT A. *Let* $K = \{h \in H;\ \| P^n h \| = \| P^{*n} h \| = \| h \|,\ n \geqq 1\}$. *Then*:

1) *K is closed subspace of H, invariant under P and P^*; P is a unitary operator on K.*

2) $h \in K \Rightarrow \operatorname{Re} h, \operatorname{Im} h \in K$; *$h \in K$ and h is real valued* $\Rightarrow h^+, h^- \in K$.

3) $Ph = e^{i\theta} h,\ h \in H \Rightarrow P|h| = |h|$

These results are from [1].

RESULT B. *P is weakly mixing if and only if the number* 1 *is the only proper value of P (considered as an operator on H) and the constants are the only proper functions.*

This is a part of the Mixing Theorem of Halmos [4, p. 39] for the operator-theoretic case. Halmos' proof of the sufficiency applies to this case also, as do Foguel's arguments in [2] which prove necessity.

LEMMA 2.1. *Let P be an ergodic and conservative Markov Process. If P^k is not ergodic for some $k > 1$, then the following holds*:

i) *$A, B \in \Sigma$ can be found such that $\lambda(A) \geqq \frac{1}{2}$, $\lambda(B) > 0$ and the density of* $M = \{n\,;\ \langle P^n \mathbf{1}_A, \mathbf{1}_B \rangle = 0\}$ *is* $\geqq \frac{1}{3}$.

ii) *$A \in \Sigma$ can be found such that $\lambda(A) > \frac{1}{4}$ and* $M = \{n\,;\ \langle P^n \mathbf{1}_A, \mathbf{1}_A \rangle = 0\}$ *is of positive density.*

PROOF. We use the following: $\Sigma_i(P^k) = \{A_1, \cdots, A_r\}$ where the sets A_i are disjoint, $\bigcup_{i=1}^{r} A_i = X$, r divides k and $P\mathbf{1}_{A_1} = \mathbf{1}_{A_2}, P\mathbf{1}_{A_2} = \mathbf{1}_{A_3}, \cdots, P\mathbf{1}_{A_r} = \mathbf{1}_{A_1}$. This is valid in a more general case than the one considered here; see [5, Th. 1].

If P^k is not ergodic then $r \geqq 2$. In the proof of (i) consider separately the cases where r is even or odd. For r even, define $A = \bigcup_{i=1}^{r/2} A_i$, $B = A_r$. Since $\lambda = \lambda P$ $\lambda(A_1) = \cdots = \lambda(A_r)$, implying that $\lambda(A_i) = 1/r$, $i = 1, \cdots, r$, so that $\lambda(A) = \frac{1}{2}$. It s easy to check that $M = \{jr, jr+1, \cdots, jr + r/2 - 1\,;\, j = 0, 1, \cdots\}$ and its density is $r/2r = \frac{1}{2}$. For r odd, define $A = \bigcup_{i=1}^{(r+1)/2}$, $B = A_r$. Here $\lambda(A) > \frac{1}{2}$ and $M = \{jr, jr+1, \cdots, jr + (r-3)/2\,;\, j = 0, 1 \cdots\}$ with density $(r-1)/2r \geqq 1/3)$ (since $r \geqq 3$). To prove (ii) take $A = \bigcup_{i=1}^{r-1} A_i$ and we are done.

REMARK. The parts of this lemma are merely steps in the proof of the two theorems to come. Obviously (ii) could be greatly improved.

THEOREM 2.1. *P is weakly mixing if and only if for every $A, B \in \Sigma$ such that $\lambda(A) \geqq \frac{1}{2}$, $\lambda(B) > 0$ there is a set of positive integers M, with lower density $< \frac{1}{4}$, and $< P^n \mathbf{1}_A, \mathbf{1}_B > > 0$ for $n \notin M$.*

Lower density is defined as $\liminf_{N \to \infty} \frac{\sigma(N)}{N}$ where $\sigma(N)$ is the number of elements in $M \cap \{1, \cdots, N\}$.

PROOF. Necessity is immediate from the definition of weak mixing and the characterization of strong Cesaro convergence by convergence outside a set of zero density: for $A, B \in \Sigma$ there exists a set M of positive integers with zero density such that $\lim_{n \notin M} \left| \langle P^n \mathbf{1}_A, \mathbf{1}_B \rangle - \lambda(A)\lambda(B) \right| = 0$.

Sufficiency will be shown by deriving a contradiction from the assumption that the condition in Result B does not hold. To begin with, P^k is ergodic for all k.

To apply Lemma 2.1 it is enough to show that P is ergodic (a process with a finite invariant measure is conservative). Indeed, assuming that $P\mathbf{1}_A = \mathbf{1}_A$ for A nontrivial, either A or its complement has measure $\geqq \frac{1}{2}$, say A, and we have $\langle P^n \mathbf{1}_A, \mathbf{1}_A \rangle = 0$ for all n. Now P cannot have proper functions for the proper value 1 other than the constants. Note that $Ph = h \Rightarrow P|h| = |h|$ (by result A), so from ergodicity, $|h| =$ constant and therefore $\operatorname{Re} h, \operatorname{Im} h \in L_\infty$. Again, by ergodicity, $\operatorname{Re} h, \operatorname{Im} h =$ constant (remember that P carries real valued functions to real valued functions).

We are led, therefore, to assume the existence of $f, g \in L_2$ and $0 < \theta < 2\pi$, such that $P^n(f + ig) = e^{in\theta}(f + ig)$ $n = 1, 2 \cdots$. According to Result A, $P|f + ig| = |f + ig|$ implying, since P is ergodic, that $f^2 + g^2 =$ a constant, and we shall assume that $f^2 + g^2 = 1$. Also, since $f + ig \in K, f^+, f^-, g^+, g^- \in K$. Note that the rotation on the unit circle by θ is irrational, since otherwise, $e^{i\theta}$ would be a root of unity, say $e^{ik\theta}= 1$, and the ergodicity of P^k would imply $\theta= 0$. We also need the following observation: for $h\in K$ and real valued, $P^nh^+ = (P^nh)^+$, and $P^nh^- = (P^nh)^-$.

Since P is unitary on K, $\langle P^nh^+, P^nh^-\rangle = 0$ so that P^nh^+, P^nh^- have disjoint supports. Writting $P^nh = P^nh^+ - P^nh^- = (P^nh)^+ - (P^nh)^-$ we have the desired equalities.

Now

$$P^nf = f\cos n\theta - g\sin n\theta$$

$$P^ng = f\sin n\theta + g\cos n\theta$$

and we consider the following differences of positive functions:

$$P^nf = f^+\cos n\theta + g^-\sin n\theta - (f^-\cos n\theta + g^+\sin n\theta)$$

$$P^ng = f^+\sin n\theta + g^+\cos n\theta - (f^-\sin n\theta + g^-\cos n\theta)$$

where $n\theta\in I = \{s \geqq 0\,;\, 0 \leqq s(\mathrm{mod}\ 2\pi) \leqq \pi/2\}$
and

$$P^mf = -f^-\cos m\theta - g^+\sin m\theta - (-f^+\cos m\theta - g^-\sin m\theta)$$

$$P^mg = -f^-\sin m\theta - g^-\cos m\theta - (-f^+\sin m\theta - g^+\cos m\theta)$$

where $m\theta\in III = \{s \geqq 0 : \pi \leqq s(\mathrm{mod}\ 2\pi) \leqq 3/2\pi\}$.

Let the supports of f^+, f^-, g^+, g^- be F^+, F^-, G^+, G^- respectively; then, using the above observation, certain inequalities can be obtained for the characteristic functions of those sets. Since P^nf^+, P^nf^- are the minimal non-negative functions such that $P^nf = P^nf^+ - P^nf^-$, one has, for $n\theta\in I$

$$P^nf^+ \leqq f^+\cos n\theta + g^-\sin n\theta,\ P^nf^- \leqq f^-\cos n\theta + g^+\sin n\theta.$$

Now, fix $\varepsilon > 0$ and let $A = \{x; f^+(x) \geqq \varepsilon\}$. Then $\varepsilon\mathbf{1}_A \leqq f^+$ and therefore, for $n\theta\in I$, $\varepsilon P^n\mathbf{1}_A \leqq P^nf^+ \leqq f^+\cos n\theta + g^-\sin n\,\theta$. Thus, $x\notin F^+\cup G^- \Rightarrow P^n\mathbf{1}_A(x)=0$; since $P^n\mathbf{1}_A \leqq \mathbf{1}$ ($\|P\| = 1$), we have $P^n\mathbf{1}_A \leqq \mathbf{1}_{F^+\cup G^-}$. But $\mathbf{1}_A$ increases pointwise towards $\mathbf{1}_{F^+}$ with ε decreasing, and P^n preserves limits of increasing sequences of non-negative functions [3, p. 4]; therefore $P^n\mathbf{1}_{F^+} \leqq \mathbf{1}_{F^+\cup G^-}$. Similarly, the corresponding inequality for $P^n\mathbf{1}_{F^-}$ can be obtained.

For $n\theta\in I$

$$P^n\mathbf{1}_{F^+} \leqq \mathbf{1}_{F^+\cup G^-}\ ,\quad P^n\mathbf{1}_{F^-} \leqq \mathbf{1}_{F^-\cup G^+}$$

$$P^n\mathbf{1}_{G^+} \leqq \mathbf{1}_{F^+\cup G^+}\ ,\quad P^n\mathbf{1}_{G^-} \leqq \mathbf{1}_{F^-\cup G^-}\,;$$

for $m\theta \in III$

$$P^m\mathbf{1}_{F^+} \leqq \mathbf{1}_{F^- \cup G^+} \quad , \quad P^m\mathbf{1}_{F^-} \leqq \mathbf{1}_{F^+ \cup G^-}$$

$$P^m\mathbf{1}_{G^+} \leqq \mathbf{1}_{F^- \cup G^-} \quad , \quad P^m\mathbf{1}_{G^-} \leqq \mathbf{1}_{F^+ \cup G^+} .$$

Now, $f^2 + g^2 = 1$ implies that $X = (F^+ \cup G^+) \cup (F^- \cup G^-)$. We shall assume that $\lambda(F^+ \cup G^+) \geqq \frac{1}{2}$; otherwise, $\lambda(F^- \cup G^-) \geqq \frac{1}{2}$ and the proof of the theorem could have been carried through for $-f-ig$ (which fulfils $P(-f-ig) = e^{i\theta}(-f-ig)$). Using the inequalities above, for $n\theta \in I$

$$P^n\mathbf{1}_{F^+ \cup G^+} \leqq P^n(\mathbf{1}_{F^+} + \mathbf{1}_{G^+}) \leqq \mathbf{1}_{F^+ \cup G^-} + \mathbf{1}_{F^+ \cup G^+}$$

so that $P^n\mathbf{1}_{F^+ \cup G^+}(x) = 0 \qquad x \notin F^+ \cup G^+ \cup G^-$
while for $m\theta \in III$

$$P^m\mathbf{1}_{F^+ \cup G^+} \leqq P^m(\mathbf{1}_{F^+} + \mathbf{1}_{G^+}) \leqq \mathbf{1}_{F^- \cup G^+} + \mathbf{1}_{F^- \cup G^-}$$

so that $P^m\mathbf{1}_{F^+ \cup G^+}(x) = 0 \qquad x \notin F^- \cup G^+ \cup G^-$.

Let $I' = \{k\,;\, k\theta \in I\}$, $III' = \{k\,;\, k\theta \in III\}$.

Note that the rotation by θ on the unit circle is an ergodic measure preserving transformation .By the Ergodic Theorem, the number of elements in $I' \cap \{1,\cdots n\}$ divided by n, approaches $\frac{1}{4}$ as $n \to \infty$(measure of the arc $[0, \pi/2]$, assuming that the measure on the circle is normalized). But this is exactly the density of I'; the same holds for III'. By the assumption in the theorem

$$F^+ \cup G^+ \cup G^- = X, \quad F^- \cup G^+ \cup G^- = X.$$

Taking intersections, this implies $G^+ \cup G^- = X$.

We check two cases: $\lambda(G^+) \geqq \frac{1}{2}$ and $\lambda(G^-) \geqq \frac{1}{2}$. Consider first $\lambda(G^+) \geqq \frac{1}{2}$; $P^n\mathbf{1}_{G^+} \leqq \mathbf{1}_{F^+ \cup G^+}$ for $n\theta \in I$, while $P^m\mathbf{1}_{G^+} \leqq \mathbf{1}_{F^- \cup G^-}$ for $m\theta \in III$.

Applying the same argument as before: $F^+ \cup G^+ = X$, $F^- \cup G^- = X$. If $\lambda(G^-) \geqq \frac{1}{2}$, the same relations will be obtained: $P^n\mathbf{1}_{G^-} \leqq \mathbf{1}_{F^- \cup G^-}$ for $n\theta \in I$, $P^m\mathbf{1}_{G^-} \leqq \mathbf{1}_{F^+ \cup G^+}$ for $m\theta \in III$. Now

$$F^+ = F^+ \cap (F^- \cup G^-) = F^+ \cap G^-, \quad G^- = G^- \cap (F^+ \cup G^+) = G^- \cap F^+$$

so that $F^+ = G^-$;

$$F^- = F^- \cap (F^+ \cup G^+) = F^- \cap G^+, \quad G^+ = G^+ \cap (F^- \cup G^-) = G^+ \cap F^-$$

so that $F^- = G^+$.

Let us summarize:

$$F^+ = G^-, \; F^- = G^+, \; G^+ \cup G^- = F^+ \cup F^- = X.$$

Finally, we again check two cases and derive a contradiction from each: $\lambda(F^+) \geqq \frac{1}{2}$ and $\lambda(F^-) \geqq \frac{1}{2}$. If $\lambda(F^+) \geqq \frac{1}{2}$ then, since $P^n\mathbf{1}_{F^+} \leqq \mathbf{1}_{F^+ \cup G^-}$ for $n\theta \in I$ and $P^m\mathbf{1}_{F^+} \leqq \mathbf{1}_{F^- \cup G^+}$ for $m\theta \in III$, by the same reasoning used above $F^+ \cup G^- = F^- \cup G^+ = X$. That is $F^+ = F^- = X$. While, if $\lambda(F^-) \geqq \frac{1}{2}$, one gets, in exactly the same way, $F^- \cup G^+ = F^+ \cup G^- = X$. This completes the proof of the theorem.

THEOREM 2.2. *P is weakly mixing if and only if P is ergodic and for every $A \in \Sigma$ such that $\lambda(A) > \frac{1}{4}$ there is a set of positive integers M, of zero lower density and*

$$\liminf_{\substack{n \notin M \\ n \to \infty}} \langle P^n\mathbf{1}_A, \mathbf{1}_A\rangle > 0.$$

PROOF. Necessity is immediate (see Theorem 2.1). Note that by Lemma 2.1, P^k is ergodic for every k. Thus we are in the same situation as in the proof of the former theorem. To prove sufficiency, we use the notations and definitions of Lemma 2.1.

We start off by showing that at least one of the sets F^+, F^-, G^+, G^- has measure greater than $\frac{1}{4}$. Indeed, since $f^2 + g^2 = 1$, $F^+ \cup F^- \cup G^+ \cup G^- = X$, and if the above is not true, these sets are mutually disjoint, each having measure equal to $\frac{1}{4}$. Therefore f^+, f^-, g^+, g^- are characteristics of their supports and for $n\theta \in I$

$$P^nf = f^+\cos n\theta + g^-\sin n\theta - (f^-\cos n\theta + g^+\sin n\theta)$$

coincides with $P^nf = P^nf^+ - P^nf^-$. Thus

$$P^n\mathbf{1}_{F^+} = \mathbf{1}_{F^+}\cos\ n\theta + \mathbf{1}_{G^-}\sin\ n\theta$$

so that

$$\tfrac{1}{4} = \langle \mathbf{1}, P^n\mathbf{1}_{F^+}\rangle = \langle \mathbf{1}, \mathbf{1}_{F^+}\cos n\theta\rangle + \langle \mathbf{1}, \mathbf{1}_{G^-}\sin n\theta\rangle = \tfrac{1}{4}\cos n\theta + \tfrac{1}{4}\sin n\theta.$$

Assume $\lambda(F^+) > \frac{1}{4}$. Otherwise, if we assume $\lambda(G^+) > \frac{1}{4}$, the proof of the theorem can be carried through for $g - if$, for which: $P(g - if) = -iP(f + ig) = e^{i\theta}(g - if)$. The cases $\lambda(F^-) > \frac{1}{4}$, $\lambda(G^-) > \frac{1}{4}$ are treated similarly.

Now, let $\varepsilon > 0$, $A \in \Sigma$ fufill $A \subset F^+$, $\lambda(A) > \frac{1}{4}$ and $\varepsilon\mathbf{1}_A \leqq f^+$. Let M be as in the statement of the theorem and $\alpha = \liminf_{n \notin M} \langle P^n\mathbf{1}_A, \mathbf{1}_A\rangle$. For $m\theta \in III$, $P^mf^+ \leqq -f^-\cos m\theta - g^+\sin m\theta$ (see the proof of Theorem 2.1). Then

$$\langle P^m\mathbf{1}_A, \mathbf{1}_A\rangle \leqq \frac{1}{\varepsilon^2}\langle P^mf^+, f^+\rangle \leqq -\frac{\sin m\theta}{\varepsilon^2}\ \langle g^+, f^+\rangle.$$

The set $M_1 = \{n \notin M;\ \langle P^n \mathbf{1}_A, \mathbf{1}_A \rangle < \alpha/2\}$ is finite and therefore $M' = M \cup M_1$ is of zero density, and we have

$$- \sin m\,\theta > \frac{\varepsilon^2 \cdot \dfrac{\alpha}{2}}{\langle g^+, f^+ \rangle}$$

for $m\theta \in III$, $m \notin M'$. A certain arc $[\pi, \pi + \beta]$, $\beta > 0$, has been obtained such that $m\theta (\mathrm{mod}\, 2\pi) \in [\pi, \pi + \beta]$ only for m in a set of zero lower density. This contradicts the irrationality of the rotation by θ.

REMARK. Analogous results can be established for a continuous time Markov Process, that is, a semi-group of Markov operators $\{P_t\}_{t \geqq 0}$ defined on $L_1(X, \Sigma, \lambda)$ which is strongly continuous in t. The proof of Result B can be adapted to yield a similar characterization in the continuous case; a proper value for $\{P_t\}$ would be a real number α such that $P_t f = e^{i\alpha t} f$ for some $0 \neq f \in H$. Proofs of the theorems then read the same as in the discrete case, with slight modifications.

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